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Noncommutative differential forms on the kappa-deformed space

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Abstract

We construct a differential algebra of forms on the kappa-deformed space. For a given realization of noncommutative coordinates as formal power series in the Weyl algebra we find an infinite family of one-forms and nilpotent exterior derivatives. We derive explicit expressions for the exterior derivative and one-forms in covariant and noncovariant realizations. We also introduce higher order forms and show that the exterior derivative satisfies the graded Leibniz rule. The differential forms are generally not graded commutative, but they satisfy the graded Jacobi identity. We also consider the star-product of classical differential forms. The star-product is well defined if the commutator between the noncommutative coordinates and one-forms is closed in the space of one-forms alone. In addition, we show that in certain realizations the exterior derivative acting on the star-product satisfies the undeformed Leibniz rule.

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1. Introduction

Recent years have witnessed a growing interest in the formulation of physical theories on noncommutative (NC) spaces. The structure of NC spaces and their physical implications was studied in [1–7]. Such spaces have roots in quantum mechanics where the canonical phase space becomes noncommutative (see [8] for a historical treatment and references therein). Classification of the NC spaces and investigation of their properties, in particular the development of a general theory suitable for physical applications, is an important problem. In this paper, we investigate differential calculus in the Euclidean kappa-deformed space. The kappa-space is a mild deformation of the Euclidean space whose coordinates $\hat{x}_\mu, \mu = 1, 2, \ldots, n$, satisfy a Lie algebra type commutation relations. The commutation relations for $\hat{x}_\mu$ depend on a deformation vector $a \in \mathbb{R}^n$ which is on a very small length scale and yields the undeformed space when $a \to 0$. The kappa-space was studied by different groups, from both the mathematical and physical points of view [9–33]. It provides a framework for doubly special relativity [18, 19], and it has applications in quantum gravity [34] and quantum field theory [35, 36].

A crucial tool in the development of a physical theory is differential calculus. There have been several attempts to develop differential calculus in the kappa-deformed space [14, 25]. For a general associative algebra Landi gave a construction of a differential algebra of forms in [37]. In this work, we present a construction of differential forms and exterior derivative in the kappa-deformed space using realizations of the NC coordinates $\hat{x}_\mu$ as formal power series in the Weyl algebra. Our approach is based on the methods developed for algebras of deformed oscillators and the corresponding creation and annihilation operators [38–47]. The realizations of the NC coordinates $\hat{x}_\mu$ in various orderings have been found in [26, 28]. The realization of a general Lie algebra type NC space in the symmetric Weyl ordering has been given in [48].

The outline of the paper is as follows. In section 2, we present a novel construction of a differential algebra of forms on the kappa-deformed space. The exterior derivative $\hat{d}$ and one-forms ξ_μ are defined as formal power series in the Lie superalgebra generated by commutative coordinates x_μ, derivatives ∂_μ and ordinary one-forms $\mathrm{d}x_\mu$. The number of one-forms ξ_μ is the same as the number of NC coordinates $\hat{x}_\mu$, and the results are valid for a general deformation vector $a \in \mathbb{R}^n$. In the present work, we do not require compatibility of the differential structure with a kappa-deformed symmetry. This distinguishes our approach from [14] where compatibility of the differential calculus with the kappa-deformed symmetry group was considered. This compatibility requires that in addition to ξ_μ there is an extra one-form ϕ. The realizations of $\hat{d}$ and ξ_μ are related to realizations of $\hat{x}_\mu$ through a system of partial differential equations. We also define higher order forms and show that $\hat{d}$ is a nilpotent operator which satisfies the graded Leibniz rule. However, the differential forms are generally not graded commutative. In the smooth limit when $a \to 0$ our theory reduces to classical results. In section 3, we analyze the exterior derivative and one-forms in covariant realizations of the kappa-deformed space. We show that the algebra generated by $\hat{x}_\mu$ and ξ_μ generally does not close under the commutator bracket since $[\xi_\mu, \hat{x}_\nu]$ may involve an infinite series in derivatives ∂_μ. We have derived a condition for the commutator $[\xi_\mu, \hat{x}_\nu]$ to be closed and found realizations in which the condition holds. A similar analysis was carried out by Dimitrijević *et al* in [25], but our results are more general and in certain aspects different. Section 4 deals with the differential algebra of forms in noncovariant realizations. We introduce a general ansatz for the exterior derivative and find the corresponding one-forms in the left, right and symmetric left–right realization. In these realizations the commutator $[\xi_\mu, \hat{x}_\nu]$ is always closed in the space of one-forms ξ_μ alone. In section 5, we present a novel construction of the star-product of (classical) differential forms. The star-product depends on realizations of $\hat{x}_\mu$ and is well defined if the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the space of one-forms ξ_μ alone. We show that for differential forms with constant coefficients the star-product is undeformed and graded commutative. However, this property does not hold for arbitrary forms. Also, we consider the induced exterior derivative acting on the star-product of differential forms. A short conclusion is given in section 6.

2. Differential forms

In this section, we present a general construction of a differential algebra of forms in the Euclidean kappa-deformed space. This construction is based on realizations of the NC coordinates $\hat{x}_\mu$ as formal power series in the Weyl algebra introduced in [26, 28]. We find that for a given realization of $\hat{x}_\mu$ there is an infinite family of exterior derivatives $\hat{d}$ and one-forms ξ_μ where ξ_μ are obtained by the action of $\hat{d}$ on $\hat{x}_\mu$. This infinite family includes two canonical types of $\hat{d}$ and ξ_μ whose realizations are studied in detail in the following sections.

The n-dimensional kappa-deformed space is a noncommutative space of Lie algebra type with generators $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \qquad a_\mu \in \mathbb{R}. \tag{1}$$

The vector $a \in \mathbb{R}^n$ describes the deformation of the n-dimensional Euclidean space. The Lie algebra satisfying (1) will be denoted by $\mathfrak{g}$. The structure constants of $\mathfrak{g}$ are given by

$$C_{\mu\nu\lambda} = a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}. \tag{2}$$

Our construction of the differential calculus uses realizations of $\hat{x}_\mu$ as formal power series in the deformation parameter a with coefficients in the Weyl algebra. The Weyl algebra is generated by the operators x_μ and ∂_μ, $\mu = 1, 2, \ldots, n$, satisfying $[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0$ and $[\partial_\mu, x_\nu] = \delta_{\mu\nu}$. It has been shown in [26, 28] that there exist infinitely many realizations of $\hat{x}_\mu$ of the form

$$\hat{x}_\mu = \sum_\alpha x_\alpha \phi_{\alpha\mu}(\partial), \tag{3}$$

where $\phi_{\alpha\mu}$ is a formal power series

$$\phi_{\alpha\mu}(\partial) = \delta_{\alpha\mu} + \sum_{|k|\geqslant 1} c_k a^{|k|} \partial^k. \tag{4}$$

We denote $\partial^k = \partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_n}$ where k is a multi-index of length $|k| = \sum_\mu k_\mu$. In the limit as $a \to 0$ we have $\phi_{\alpha\mu} \to \delta_{\alpha\mu}$, whence $\hat{x}_\mu$ become the commutative coordinates x_μ. A representation (3) of the NC coordinates $\hat{x}_\mu$ will be called a ϕ-realization. The NC coordinates $\hat{x}_\mu$ and derivatives ∂_μ generate a deformed Heisenberg algebra satisfying

$$[\partial_\mu, \hat{x}_\nu] = \phi_{\mu\nu}(\partial). \tag{5}$$

We will assume that the matrix $[\phi_{\mu\nu}]$ is invertible, allowing us to express x_μ as

$$x_\mu = \sum_\alpha \hat{x}_\alpha \phi^{-1}_{\alpha\mu}(\partial), \tag{6}$$

where $\phi^{-1}_{\alpha\mu}(\partial)$ is also a formal power series of the type (4). The existence of $\phi^{-1}_{\mu\nu}$ implies that there is a vector space isomorphism between the symmetric algebra generated by x_μ, $\mu = 1, 2, \ldots, n$, and the enveloping algebra of $\mathfrak{g}$. This isomorphism will be important in defining the star-product discussed in section 5. With regard to the action of the rotation algebra $so(n)$ the realizations of the kappa-space can be divided into covariant [28] and noncovariant [26]. Both types of realizations will be used in the construction of differential forms in sections 3 and 4.

It is useful to introduce a unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ generated by x_μ, ∂_μ and ordinary one-forms $\mathrm{d}x_\mu$, $1 \leqslant \mu \leqslant n$, satisfying the additional relations $[\mathrm{d}x_\mu, x_\nu] = [\mathrm{d}x_\mu, \partial_\nu] = 0$ and $\{\mathrm{d}x_\mu, \mathrm{d}x_\nu\} = 0$ where $\{,\}$ denotes the anticommutator. A basis for $\mathcal{A}$ consists of the monomials

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}\, \mathrm{d}x_{\sigma_1} \cdots \mathrm{d}x_{\sigma_p}, \tag{7}$$

where $\alpha_i, \beta_i \in \mathbb{N}_0$ and $1 \leqslant \sigma_1 < \sigma_2 \cdots < \sigma_p \leqslant n$ for $p = 1, 2, \ldots, n$. We define a $\mathbb{Z}_2$-gradation of $\mathcal{A}$ by $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0$ and $\mathcal{A}_1$ are spanned by the monomials (7) with p even and odd, respectively. The algebra $\mathcal{A}$ is equipped with the graded commutator defined on homogeneous elements by

$$[[u, v]] = uv - (-1)^{|u||v|} vu, \tag{8}$$

where $|u|$ denotes the degree of u, ($|u| = 0$ or $|u| = 1$). The commutator (8) makes $\mathcal{A}$ into a Lie superalgebra, and it satisfies the graded Jacobi identity

$$(-1)^{|u||w|}[[u, [[v, w]]]] + (-1)^{|v||u|}[[v, [[w, u]]]] + (-1)^{|w||v|}[[w, [[u, v]]]] = 0. \tag{9}$$

Recall that in the ordinary Euclidean space the exterior derivative is given by $d = \sum_\alpha \mathrm{d}x_\alpha \partial_\alpha$. It is a nilpotent operator, $d^2 = 0$, satisfying the commutation relation $[d, x_\mu] = \mathrm{d}x_\mu$. Our goal is to construct smooth deformations of d and $\mathrm{d}x_\mu$, denoted by $\hat{d}$ and ξ_μ, $\mu = 1, 2, \ldots, n$, which preserve the basic relation

$$[\hat{d}, \hat{x}_\mu] = \xi_\mu. \tag{10}$$

Let us assume that $\hat{d}$ and ξ_μ are represented by

$$\xi_\mu = \sum_\alpha \mathrm{d}x_\alpha h_{\alpha\mu}(\partial) \qquad \text{and} \qquad \hat{d} = \sum_{\alpha,\beta} \mathrm{d}x_\alpha \partial_\beta k_{\alpha\beta}(\partial), \tag{11}$$

where $h_{\mu\nu}$ and $k_{\mu\nu}$ are formal power series of the type (4). The boundary conditions $\lim_{a\to 0} h_{\mu\nu} = \delta_{\mu\nu}$ and $\lim_{a\to 0} k_{\mu\nu} = \delta_{\mu\nu}$ ensure that in the smooth limit $\xi_\mu \to \mathrm{d}x_\mu$ and $\hat{d} \to d$ as $a \to 0$. As in the classical case, the deformed one-forms anticommute and the exterior derivative is nilpotent. Indeed,

$$\{\xi_\mu, \xi_\nu\} = \sum_{\alpha<\beta} \{\mathrm{d}x_\alpha, \mathrm{d}x_\beta\}(h_{\alpha\mu}h_{\beta\nu} + h_{\alpha\nu}h_{\beta\mu}) = 0, \tag{12}$$

$$\hat{d}^2 = \sum_{\alpha<\beta} \{\mathrm{d}x_\alpha, \mathrm{d}x_\beta\} \sum_{\mu,\nu} \partial_\mu \partial_\nu k_{\alpha\mu} k_{\beta\nu} = 0, \tag{13}$$

since $\{\mathrm{d}x_\alpha, \mathrm{d}x_\beta\} = 0$. We assume that the matrix $[h_{\mu\nu}]$ is invertible so that we may express $\mathrm{d}x_\mu$ in terms of ξ_μ. Using representation (11) one finds that the commutation relation (10) is equivalent to a system of partial differential equations for the unknown functions $h_{\mu\nu}$ and $k_{\mu\nu}$:

$$\sum_\rho \left(k_{\alpha\rho} + \sum_\beta \frac{\partial k_{\alpha\beta}}{\partial \partial_\rho} \partial_\beta \right) \phi_{\rho\mu} = h_{\alpha\mu}. \tag{14}$$

This is an underdetermined system of n^2 equations for $2n^2$ unknown functions. Taking the commutator of $\hat{d}$ with both sides of the commutation relations (1) and applying the Jacobi identity to the commutator $[\hat{d}, [\hat{x}_\mu, \hat{x}_\nu]]$, we find that $\hat{x}_\mu$ and ξ_ν satisfy the compatibility condition

$$[\hat{x}_\mu, \xi_\nu] - [\hat{x}_\nu, \xi_\mu] = \mathrm{i}(a_\mu \xi_\nu - a_\nu \xi_\mu). \tag{15}$$

Hence, every solution of equation (14) must be compatible with the differential equation implicit in (15). We note that equation (15) implies that since $a \neq 0$, not all commutators $[\hat{x}_\mu, \xi_\nu]$ can be simultaneously zero.

The condition (15) places constraints on the choice of $k_{\mu\nu}$ and $h_{\mu\nu}$. For a given function $k_{\mu\nu}$ satisfying $\lim_{a\to 0} k_{\mu\nu} = \delta_{\mu\nu}$, equation (14) uniquely determines $h_{\mu\nu}$. The boundary conditions imposed on $\phi_{\mu\nu}$ and $k_{\mu\nu}$ imply that $\lim_{a\to 0} h_{\alpha\mu} = \delta_{\alpha\mu}$ automatically holds. Therefore, starting with the exterior derivative $\hat{d}$ one readily finds the one-forms ξ_μ satisfying equation (10). However, the converse is not true since one cannot always find $k_{\mu\nu}$ for an arbitrary choice of $h_{\mu\nu}$. For example, if $h_{\mu\nu} = \delta_{\mu\nu}$ then equation (11) implies that ξ_μ is the ordinary one-form, $\xi_\mu = \mathrm{d}x_\mu$. In this case $[\hat{x}_\mu, \xi_\nu] = 0$ for all $\mu, \nu = 1, 2, \ldots n$, which contradicts the compatibility condition (15).

Let $\bar{\mathcal{A}}$ denote the formal completion of $\mathcal{A}$. We associate with the exterior derivative $\hat{d}$ a linear map or action $\hat{d}\colon \bar{\mathcal{A}} \to \bar{\mathcal{A}}$ defined by

$$\hat{d} \cdot u = [[\hat{d}, u]]. \tag{16}$$

It follows from equation (10) that $\hat{d} \cdot \hat{x}_\mu = \xi_\mu$, hence the action of $\hat{d}$ on the coordinate $\hat{x}_\mu$ yields the one-form ξ_μ. The action of $\hat{d}$ on the product of homogeneous elements $u, v, \in \bar{\mathcal{A}}$ satisfies the graded Leibniz rule

$$\hat{d} \cdot (uv) = (\hat{d} \cdot u)v + (-1)^{|u|} u(\hat{d} \cdot v). \tag{17}$$

For zero-forms $\hat{f} = \hat{f}(\hat{x})$ and $\hat{g} = \hat{g}(\hat{x})$ this reduces to the undeformed Leibniz rule

$$\hat{d} \cdot (\hat{f}\hat{g}) = (\hat{d} \cdot \hat{f})\hat{g} + \hat{f}(\hat{d} \cdot \hat{g}). \tag{18}$$

It turns out that it is quite natural to consider the following canonical representation of $\hat{d}$ and ξ_μ:

Type I

$$\hat{d} = \sum_\alpha \mathrm{d}x_\alpha \partial_\alpha, \qquad \xi_\mu = \sum_\alpha \mathrm{d}x_\alpha \phi_{\alpha\mu}(\partial), \tag{19}$$

Type II

$$\hat{d} = \sum_\alpha \xi_\alpha \partial_\alpha, \qquad \xi_\mu = \sum_\alpha \mathrm{d}x_\alpha h_{\alpha\mu}(\partial). \tag{20}$$

The first type is obtained by choosing $k_{\mu\nu} = \delta_{\mu\nu}$, in which case equation (14) yields $h_{\mu\nu} = \phi_{\mu\nu}$. This provides the simplest possible realization of the one-form ξ_μ. The second type is obtained by demanding that $k_{\mu\nu} = h_{\mu\nu}$. Then the functions $h_{\mu\nu}$ satisfy the system of partial differential equations

$$\sum_\rho \left(h_{\alpha\rho} + \sum_\beta \frac{\partial h_{\alpha\beta}}{\partial \partial_\rho} \partial_\beta \right) \phi_{\rho\mu} = h_{\alpha\mu} \tag{21}$$

subject to the boundary conditions $\lim_{a\to 0} h_{\mu\nu} = \delta_{\mu\nu}$. In this case both the exterior derivative $\hat{d}$ and one-forms ξ_μ depend in a very nontrivial manner on the given ϕ-realization. In the following sections, we shall analyze $\hat{d}$ and ξ_μ in covariant and noncovariant realizations found in [26, 28]. Note that the generators $\hat{x}_\mu, \partial_\mu, \xi_\mu, 1 \leqslant \mu \leqslant n$, form an associative superalgebra which inherits the grading from the superalgebra $\mathcal{A}$. The subalgebra generated by $\hat{x}_\mu, \partial_\mu, 1 \leqslant \mu \leqslant n$ is the deformed Heisenberg algebra (5).

So far we have defined the exterior derivative $\hat{d}$ and one-forms ξ_μ such that $\hat{d} \cdot \hat{x}_\mu = \xi_\mu$. We would like to extend the above construction to higher order forms so that the action of $\hat{d}$ on k-forms yields $(k+1)$-forms. First, we need to define what is meant by a k-form for $k \geqslant 1$. A k-form is a finite linear combination of monomials in $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ and $\xi_1, \xi_2, \dots, \xi_n$ such that there are precisely k one-forms ξ_μ in each monomial. The one-forms ξ_μ may be placed in any order in a given monomial. For example, both $\hat{\omega}^1 = \hat{x}_\mu \hat{x}_\nu \xi_\rho$ and $\hat{\eta}^1 = \hat{x}_\mu \xi_\rho \hat{x}_\nu$ are one-forms, albeit different. Let $\hat{\Omega}^k$ denote the space of k-forms and let $\hat{\Omega} = \bigoplus_{k\geqslant 0} \hat{\Omega}^k$. The multiplication in $\hat{\Omega}$ is simply given by juxtaposition of the elements. This defines a grading on $\hat{\Omega}$ since $\hat{\Omega}^k \hat{\Omega}^l \subseteq \hat{\Omega}^{k+l}$. We note that the product of differential forms is not graded commutative in general,

$$\hat{\omega}^k \hat{\eta}^l \neq (-1)^{kl} \hat{\eta}^l \hat{\omega}^k. \tag{22}$$

The product is graded commutative only for constant forms $\hat{\omega}^k = \xi_{\mu_1}\xi_{\mu_2}\cdots\xi_{\mu_k}$ since ξ_{μ_i} and ξ_{μ_j} anticommute.

Next we show that the exterior derivative $\hat{d}$ maps $\hat{\Omega}^k$ into $\hat{\Omega}^{k+1}$ for $k \geqslant 0$. First, using the Leibniz rule (17) it is easily seen that

$$\hat{d} \cdot \hat{f}(\hat{x}) \in \hat{\Omega}^1 \qquad \text{for all} \qquad \hat{f}(\hat{x}) \in \hat{\Omega}^0. \tag{23}$$

Furthermore, using equation (11) we find

$$\hat{d} \cdot \xi_\mu = [[\hat{d}, \xi_\mu]] = \hat{d}\xi_\mu + \xi_\mu \hat{d} = 0 \tag{24}$$

since $\{\mathrm{d}x_\mu, \mathrm{d}x_\nu\} = 0$. By induction on k one can show that

$$\hat{d} \cdot (\xi_{\mu_1}\xi_{\mu_2}\cdots\xi_{\mu_k}) = 0 \qquad \text{for all} \qquad k \geqslant 1. \tag{25}$$

Relations (23) and (25) together with the Leibniz rule (17) imply that $\hat{d}$ maps k-forms to $(k+1)$-forms. For example,

$$\hat{d} \cdot (\hat{x}_\mu \hat{x}_\nu \xi_\lambda) = \hat{d} \cdot (\hat{x}_\mu \hat{x}_\nu)\xi_\lambda = \xi_\mu \hat{x}_\nu \xi_\lambda + \hat{x}_\mu \xi_\nu \xi_\lambda. \tag{26}$$

The exterior derivative satisfies the graded Leibniz rule

$$\hat{d} \cdot (\hat{\omega}^k \hat{\eta}^l) = (\hat{d} \cdot \hat{\omega}^k)\hat{\eta}^l + (-1)^k \hat{\omega}^k (\hat{d} \cdot \hat{\eta}^l). \tag{27}$$

Hence, the algebra $\hat{\Omega}$ together with the linear map $\hat{d}\colon \hat{\Omega}^k \to \hat{\Omega}^{k+1}$ is a differential algebra. Our approach is essentially the same as the construction of the differential algebra of forms discussed in [37]. In our case the algebra of zero-forms has the additional structure of the universal enveloping algebra satisfying relations (1). We note that in general one cannot rewrite a given k-form such that $\xi_{\mu_1}, \xi_{\mu_2}, \ldots, \xi_{\mu_k}$ are placed to the far right. This is possible only in special realizations in which the commutator $[\xi_\mu, \hat{x}_\nu]$ closes in the space of one-forms ξ_μ alone.

3. Covariant realizations

In this section, we shall investigate the differential algebra of forms in covariant realizations of the kappa-deformed space introduced in [28]. These realizations are covariant under the action of the rotation aglebra $so(n)$. Of particular interest is a class of simple realizations obtained for the following choice of $\phi_{\mu\nu}$ in the representation (3):

Left realization:

$$\phi_{\mu\nu} = (1 - A)\delta_{\mu\nu}, \tag{28}$$

Right realization:

$$\phi_{\mu\nu} = \delta_{\mu\nu} + \mathrm{i}a_\nu \partial_\mu, \tag{29}$$

Natural realization:

$$\phi_{\mu\nu}(\partial) = (-A + \sqrt{1 - B})\delta_{\mu\nu} + \mathrm{i}a_\mu \partial_\nu, \tag{30}$$

Symmetric realization:

$$\phi_{\mu\nu} = \frac{A}{e^A - 1}\delta_{\mu\nu} + \mathrm{i}a_\nu \partial_\mu \frac{e^A - A - 1}{(e^A - 1)A}. \tag{31}$$

Here A and B are commuting operators defined by $A = \mathrm{i}a\partial$ and $B = a^2\partial^2$ where we use the convention $a\partial = \sum_\alpha a_\alpha \partial_\alpha$, $\partial^2 = \sum_\alpha \partial_\alpha^2$, etc. The symmetric realization corresponds to the Weyl symmetric ordering of the monomials in $\hat{x}_\mu$. We remark that for a general Lie algebra type NC space there is a universal formula for $\phi_{\mu\nu}$ in Weyl symmetric ordering given in [48] as follows. Suppose $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ are generators of a Lie algebra with structure constants $\theta_{\mu\nu\alpha}$:

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i} \sum_\alpha \theta_{\mu\nu\alpha} \hat{x}_\alpha. \tag{32}$$

Let $M = [M_{\mu\nu}]$ denote the $n \times n$ matrix of differential operators with elements

$$M_{\mu\nu} = \mathrm{i} \sum_\alpha \theta_{\alpha\nu\mu} \partial_\alpha. \tag{33}$$

Then the Weyl symmetric realization of the Lie algebra (32) is given by

$$\phi_{\mu\nu}(\partial) = p(M)_{\mu\nu} \qquad \text{where} \qquad p(M) = \frac{M}{e^M - 1} \tag{34}$$

is the generating function for the Bernoulli numbers (see also [49]). In principle, the exterior derivative and one-forms may be constructed using any of the above realizations. Here we shall consider the left, right and natural realizations.

3.1. Covariant realizations of type I

Let us consider realizations of type I where the exterior derivative is undeformed, $\hat{d} = \sum_\alpha \mathrm{d}x_\alpha \partial_\alpha$, and one-forms are given by $\xi_\mu = \sum_\alpha \mathrm{d}x_\alpha \phi_{\alpha\mu}(\partial)$. We investigate the conditions under which the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the space of one-forms ξ_μ. The closedness of the commuatator is important when considering the extended star-product of (classical) forms in section 5.

Using realization (3) we have

$$[\xi_\mu, \hat{x}_\nu] = \sum_\alpha \sum_\beta \mathrm{d}x_\alpha \frac{\partial \phi_{\alpha\mu}}{\partial \partial_\beta} \phi_{\beta\nu}. \tag{35}$$

The matrix $[\phi_{\mu\nu}]$ is invertible, hence we may express $\mathrm{d}x_\mu$ in terms of ξ_μ to obtain

$$[\xi_\mu, \hat{x}_\nu] = \sum_\sigma C_{\mu\nu\sigma}(\partial) \xi_\sigma, \tag{36}$$

where

$$C_{\mu\nu\sigma}(\partial) = \sum_\alpha \sum_\beta \phi^{-1}_{\sigma\alpha} \frac{\partial \phi_{\alpha\mu}}{\partial \partial_\beta} \phi_{\beta\nu}. \tag{37}$$

Clearly, the commutator (36) is closed in the space of one-forms ξ_μ only if the coefficients $C_{\mu\nu\sigma}$ are constant. This condition is satisfied in the left and right realizations, as shown in the following. In the left realization, we have

$$\hat{x}_\mu = x_\mu(1 - A), \qquad \xi_\mu = \mathrm{d}x_\mu(1 - A), \tag{38}$$

which yields

$$[\xi_\mu, \hat{x}_\nu] = -\mathrm{i} a_\nu \xi_\mu. \tag{39}$$

Similarly, in the right realization we have

$$\hat{x}_\mu = x_\mu + \mathrm{i} a_\mu (x\partial), \qquad \xi_\mu = \mathrm{d}x_\mu + \mathrm{i} a_\mu (\mathrm{d}x \partial), \tag{40}$$

which leads to

$$[\xi_\mu, \hat{x}_\nu] = \mathrm{i} a_\mu \xi_\nu. \tag{41}$$

On the other hand, in the natural and symmetric realizations the coefficients $C_{\mu\nu\sigma}$ involve partial derivatives so the commutators between ξ_μ and $\hat{x}_\nu$ are not closed.

3.2. Covariant realizations of type II

Consider now realizations of type II where the exterior derivative and one-forms are given by $\hat{d} = \sum_\alpha \xi_\alpha \partial_\alpha$ and $\xi_\mu = \sum_\alpha \mathrm{d}x_\alpha h_{\alpha\mu}(\partial)$, and $h_{\alpha\mu}$ is a solution to equation (21). In this section, we shall construct $\hat{d}$ and ξ_μ using the natural realization (30). The construction of NC forms in type II realization was considered in [25], but not in a proper and complete way. Our motivation for using the natural realization is to present a proper analysis of this problem.

Let us write equation (21) in a more compact form

$$\sum_\rho \frac{\partial \Lambda_\alpha}{\partial \partial_\rho} \phi_{\rho\mu} = h_{\alpha\mu}, \tag{42}$$

where $\Lambda_\alpha(\partial) = \sum_\beta h_{\alpha\beta}(\partial)\partial_\beta$. The idea is to solve an auxiliary problem for Λ_α and then calculate $h_{\mu\nu}$ from equation (42). Multiplying equation (42) by ∂_μ and summing we obtain the following boundary value problem for Λ_α:

$$\sum_\rho \frac{\partial \Lambda_\alpha}{\partial \partial_\rho} \Psi_\rho = \Lambda_a, \qquad \lim_{a\to 0} \Lambda_\alpha = \partial_\alpha, \tag{43}$$

where $\Psi_\rho(\partial) = \sum_\mu \phi_{\rho\mu}(\partial)\partial_\mu$. In the natural realization (30) we find

$$\Psi_\rho(\partial) = \partial_\rho(-A + \sqrt{1-B}) + \mathrm{i}a_\rho \partial^2. \tag{44}$$

Let us denote $Z^{-1} = -A + \sqrt{1-B}$. This is the inverse shift operator introduced in [28]. The index structure of Ψ_ρ and equation (43) suggest that we should look for Λ_α in the form

$$\Lambda_\alpha(\partial) = \partial_\alpha H_1(A,B) + \mathrm{i}a_\alpha \partial^2 H_2(A,B) \tag{45}$$

for unknown functions H_1 and H_2. From equations (44) and (45) we obtain

$$\begin{aligned}\sum_\rho \frac{\partial \Lambda_\alpha}{\partial \partial_\rho} \Psi_\rho = {} & \partial_\alpha \left[\left(H_1 + A\frac{\partial H_1}{\partial A} + 2B\frac{\partial H_1}{\partial B} \right) Z^{-1} - B\frac{\partial H_1}{\partial A} + 2AB\frac{\partial H_1}{\partial B} \right] \\ & + \mathrm{i}a_\alpha \partial^2 \left[\left(2H_2 + A\frac{\partial H_2}{\partial A} + 2B\frac{\partial H_2}{\partial B} \right) Z^{-1} + H_1 + 2AH_2 - B\frac{\partial H_2}{\partial A} + 2AB\frac{\partial H_2}{\partial B} \right].\end{aligned} \tag{46}$$

Substituting the above result into equation (43) we find that H_1 and H_2 satisfy the following system of differential equations:

$$\left(H_1 + A\frac{\partial H_1}{\partial A} + 2B\frac{\partial H_1}{\partial B} \right) Z^{-1} - B\frac{\partial H_1}{\partial A} + 2AB\frac{\partial H_1}{\partial B} = H_1, \tag{47}$$

$$\left(2H_2 + A\frac{\partial H_2}{\partial A} + 2B\frac{\partial H_2}{\partial B} \right) Z^{-1} - B\frac{\partial H_2}{\partial A} + 2AB\frac{\partial H_2}{\partial B} + 2AH_2 + H_1 = H_2. \tag{48}$$

Since $\Lambda_\alpha(\partial) \to \partial_a$ as $a \to 0$, H_1 and H_2 are subject to the boundary conditions

$$\lim_{a\to 0} H_1(A,B) = 1, \qquad \lim_{a\to 0} H_2(A,B) \text{ finite}. \tag{49}$$

It is shown in appendix A that the above system has a unique solution

$$H_1(A,B) = \frac{2(1-\sqrt{1-B})}{B(-A+\sqrt{1-B})}, \tag{50}$$

$$H_2(A,B) = -2(1-A+\sqrt{1-B}) \left(\frac{1-\sqrt{1-B}}{B} \right)^2. \tag{51}$$

Inserting the expressions for H_1 and H_2 into equation (45) we find

$$\Lambda_\alpha(\partial) = \partial_\alpha \frac{2(1-\sqrt{1-B})}{B(-A+\sqrt{1-B})} - \mathrm{i}a_\alpha \partial^2 2(1-A+\sqrt{1-B})\left(\frac{1-\sqrt{1-B}}{B}\right)^2. \tag{52}$$

Since the exterior derivative is given by $\hat{d} = \sum_\alpha \xi_a \partial_\alpha$ where $\xi_\mu = \sum_\alpha \mathrm{d}x_\alpha h_{\alpha\mu}(\partial)$, $\hat{d}$ can be expressed in terms of Λ_α as

$$\hat{d} = \sum_\alpha \mathrm{d}x_\alpha \Lambda_\alpha(\partial). \tag{53}$$

Thus, we find from equation (52) that

$$\hat{d} = \frac{2(1-\sqrt{1-B})}{B(-A+\sqrt{1-B})}(\partial \mathrm{d}x) - 2(1-A+\sqrt{1-B})\left(\frac{1-\sqrt{1-B}}{B}\right)^2 \mathrm{i}(a\mathrm{d}x)\partial^2. \tag{54}$$

Keeping only the first-order terms in $a \in \mathbb{R}^n$ we obtain the approximation

$$\hat{d} = \partial \mathrm{d}x + \mathrm{i}(a\partial)(\partial \mathrm{d}x) - \mathrm{i}\partial^2(a\mathrm{d}x), \tag{55}$$

where $d = \partial \mathrm{d}x$ is the undeformed exterior derivative.

Next we consider the one-form ξ_μ. Substituting equations (30) and (52) into equation (42) we find after some manipulation that

$$h_{\alpha\mu}(\partial) = L_1\delta_{\alpha\mu} + \mathrm{i}L_2 a_\alpha \partial_\mu + \mathrm{i}L_3 a_\mu \partial_a + a^2 L_4 \partial_\alpha \partial_\mu - \partial^2 L_5 a_\alpha a_\mu, \tag{56}$$

where

$$L_1 = \frac{2(1-\sqrt{1-B})}{B}, \tag{57}$$

$$L_2 = -\frac{2(-1+\sqrt{1-B})\left[2(A^2+A-B)\sqrt{1-B}+B-2(A^2-2AB+A)\right]}{B^2(-A+\sqrt{1-B})}, \tag{58}$$

$$L_3 = \frac{2(1-\sqrt{1-B})}{B(-A+\sqrt{1-B})}, \tag{59}$$

$$L_4 = -\frac{2(B+2\sqrt{1-B}-2)}{B^2(-A+\sqrt{1-B})}, \tag{60}$$

$$L_5 = \frac{2(-A+\sqrt{1-B})(1-\sqrt{1-B})^2}{B^2}. \tag{61}$$

Therefore, in the natural realization of type II the one-form ξ_μ is given by

$$\begin{aligned}\xi_\mu &= \sum_\alpha h_{\alpha\mu}(\partial)\, \mathrm{d}x_\alpha \\ &= L_1\, \mathrm{d}x_\mu + (\mathrm{i}L_2\partial_\mu - \partial^2 L_5 a_\mu)(a\mathrm{d}x) + (\mathrm{i}L_3 a_\mu + a^2 L_4 \partial_\mu)(\partial \mathrm{d}x).\end{aligned} \tag{62}$$

Although the above realization of ξ_μ is rather complicated, the first-order approximation has a particularly nice form

$$\xi_\mu = \mathrm{d}x_\mu + \sum_\alpha \mathrm{i}(a_\mu \partial_\alpha - a_\alpha \partial_\mu)\, \mathrm{d}x_\alpha. \tag{63}$$

Let us now investigate the commutation relations for ξ_μ and $\hat{x}_\nu$. The NC coordinates in the natural realization (30) are given by

$$\hat{x}_\mu = x_\mu(-A+\sqrt{1-B}) + \mathrm{i}(ax)\partial_\mu. \tag{64}$$

The explicit form of the commutator $[\xi_\mu, \hat{x}_\nu]$ is fairly complicated and a complete derivation is given in appendix B. Here we only state that it can be expressed as

$$[\xi_\mu, \hat{x}_\nu] = \xi_\mu \frac{P_\nu^{(1)}}{L_1} + \xi_\nu \frac{P_\mu^{(2)}}{L_1} + (\mathrm{i}a\xi) R_{\mu\nu}^{(1)} + (\partial\xi) R_{\mu\nu}^{(2)}, \tag{65}$$

where $P_\mu^{(i)}$ and $R_{\mu\nu}^{(i)}$ are certain combinations of the functions $L_1, L_2, \ldots, L_5$ and their partial derivatives. We note that the commutator (65) is not closed since the right-hand side involves derivatives ∂_μ. To gain an insight into the form of the commutator it is instructive to find a first-order approximation in the parameter a. To first order in a the natural realization of $\hat{x}_\mu$ is given by

$$\hat{x}_\mu = x_\mu(1 - \mathrm{i}a\partial) + \mathrm{i}(ax)\partial_\mu. \tag{66}$$

Using the approximations (63) and (66) we obtain

$$[\xi_\mu, \hat{x}_\nu] = \mathrm{i} \sum_\alpha (a_\mu \delta_{\alpha\nu} - a_\alpha \delta_{\mu\nu}) \xi_\alpha. \tag{67}$$

As a special case suppose that the vector $a \in \mathbb{R}^n$ has only one non-zero component, $a_\mu = a\delta_{\mu n}$ for $\mu = 1, 2, \ldots, n$. Then

$$[\xi_\mu, \hat{x}_\nu] = \mathrm{i}a(\delta_{\mu n}\xi_\nu - \delta_{\mu\nu}\xi_n). \tag{68}$$

The above result agrees to first order in a with the commutator $[\xi_\mu, \hat{x}_\nu]$ for vector-like transforming one-forms considered in [25]. We emphasize, however, that the exact expression (65) does not agree with this commutator for higher orders in a.

4. Noncovariant realizations

In this section, we consider the exterior derivative and one-forms in noncovariant realizations of the kappa-space introduced in [26]. We assume that the components of the deformation vector $a \in \mathbb{R}^n$ are given by $a_k = 0$ for $k = 1, 2, \ldots, n-1$ and $a_n = a$. Then the commutation relations (1) yield

$$[\hat{x}_k, \hat{x}_l] = 0, \qquad [\hat{x}_n, \hat{x}_k] = \mathrm{i}a\hat{x}_k, \qquad k, l = 1, 2, \ldots, n-1. \tag{69}$$

We use the Latin alphabet for the indices $1, 2, \ldots, n-1$ and the Greek alphabet for the full set $1, 2, \ldots, n$. It was shown in [26] that the NC coordinates $\hat{x}_\mu$ have infinitely many realizations of the form

$$\hat{x}_k = x_k \varphi(A), \qquad k = 1, 2, \ldots, n-1, \tag{70}$$

$$\hat{x}_n = x_n + \mathrm{i}a \sum_{k=1}^{n-1} x_k \partial_k \gamma(A), \tag{71}$$

where

$$\gamma(A) = \frac{\varphi'(A)}{\varphi(A)} + 1, \qquad A = \mathrm{i}a\partial_n. \tag{72}$$

The realizations are parametrized by the function $\varphi(A)$ satisfying the boundary conditions $\lim_{a\to 0} \varphi(A) = 1$ and $\lim_{a\to 0} \varphi'(A)$ finite, so that $\hat{x}_\mu \to x_\mu$ as $a \to 0$. The NC coordinates $\hat{x}_\mu$ are covariant under the rotation algebra $so(n-1)$, but not generally under the full algebra $so(n)$.

The most general ansatz for the exterior derivative $\hat{d}$ invariant under $so(n-1)$ is

$$\hat{d} = \sum_{k=1}^{n-1} \mathrm{d}x_k \partial_k N_1(A, \Delta) + \mathrm{d}x_n \partial_n N_2(A, \Delta) + \mathrm{i}a\, \mathrm{d}x_n \sum_{k=1}^{n-1} \partial_k^2 G(A, \Delta), \tag{73}$$

where $\Delta = (\mathrm{i}a)^2 \sum_{k=1}^{n-1} \partial_k^2$. The family of realizations (70) and (71) includes special realizations corresponding to the left, right, symmetric left–right and symmetric Weyl orderings for the enveloping algebra of the Lie algebra (69). These realizations are parametrized by

$$\varphi(A) = e^{-A}, \qquad \varphi(A) = 1, \qquad \varphi(A) = e^{-A/2} \qquad \text{and} \qquad \varphi(A) = A/(e^A - 1), \tag{74}$$

respectively. We remark that only the symmetric Weyl realization is covariant under the full algebra $so(n)$.

For a given parameter function φ and an arbitrary choice of N_1, N_2 and G one can find the one-forms ξ_k satisfying $[\hat{d}, \hat{x}_\mu] = \xi_\mu$. As in the case of the covariant realizations one can express the commutator $[\xi_\mu, \hat{x}_\nu]$ in terms of the one-forms ξ_μ and partial derivatives ∂_μ, but the general expressions are fairly complicated.

In the following, we will focus our attention to a subfamily of the noncovariant realizations which lead to some interesting results. These realizations are parametrized by $\varphi(A) = e^{-cA}$, $c \in \mathbb{R}$:

$$\hat{x}_k = x_k e^{-cA}, \qquad k = 1, 2, \ldots, n-1, \tag{75}$$

$$\hat{x}_n = x_n + \mathrm{i}a(1-c) \sum_{k=1}^{n-1} x_k \partial_k. \tag{76}$$

They include the left, right and symmetric left–right realizations for $c = 1$, $c = 0$ and $c = 1/2$, respectively. Let us define the exterior derivative by

$$\hat{d} = \sum_{k=1}^{n-1} \mathrm{d}x_k \partial_k\, e^{(c-1)A} + \mathrm{d}x_n \partial_n \tag{77}$$

($N_1 = e^{(c-1)A}$, $N_2 = 1$, $G = 0$). Then the corresponding one-forms are given by

$$\xi_k = [\hat{d}, \hat{x}_k] = \mathrm{d}x_k\, e^{-A}, \qquad k = 1, 2, \ldots, n-1, \tag{78}$$

$$\xi_n = [\hat{d}, \hat{x}_n] = \mathrm{d}x_n. \tag{79}$$

The algebra generated by $\hat{x}_\mu$ and ξ_μ satisfies the commutation relations

$$[\xi_k, \hat{x}_l] = 0, \qquad [\xi_k, \hat{x}_n] = -\mathrm{i}a\xi_k, \tag{80}$$

$$[\xi_n, \hat{x}_l] = 0, \qquad [\xi_n, \hat{x}_n] = 0. \tag{81}$$

This algebra satisfies the graded Jacobi relations (9). We note that relations (80) and (81) correspond to the algebra found by Kim *et al* [50] where the commutators are defined in terms of the star-product, except that in our work ξ_μ and ξ_ν anticommute. In particular, for $c = 0$ the exterior derivative becomes

$$\hat{d} = \sum_{k=1}^{n-1} \mathrm{d}x_k \partial_k e^{-A} + \mathrm{d}x_n \partial_n = \sum_{\alpha=1}^{n} \xi_\alpha \partial_\alpha, \tag{82}$$

which is the type II realization of $\hat{d}$. In addition to the examples in section 3 the commutators (80) and (81) also close in the space of one-forms ξ_μ alone. Moreover, the right realization ($c = 0$) is an example of a type II realization with closed commutator.

The above construction can be extended to any parameter function φ. It can be shown that for a given φ one can find N_1, N_2 and G such that $\hat{d} = \sum_\alpha \xi_\alpha \partial_\alpha$ and $[\hat{d}, \hat{x}_\mu] = \xi_\mu$. However, this may be very complicated as already seen in the natural realization in section 3.

5. Extended star-product

Regarding functions as zero-forms we want to extend the star-product to differential forms of arbitrary degree. The star-product of differential forms in the context of deformation quantization has been investigated recently in [51]. The construction of the star-product presented here is valid for a general Lie algebra type noncommutative space. We recall that the realization of NC coordinates $\hat{x}_\mu$ in terms of x_μ and ∂_μ is given by equation (3). Also, since the matrix $[\phi_{\mu\nu}]$ is invertible the commutative coordinates x_μ admit realization in terms of $\hat{x}_\mu$ and ∂_μ via equation (6). The duality between $\hat{x}_\mu$ and x_μ induces a vector space isomorphism $\Omega_\phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{S}$ between the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra (1) and the symmetric algebra $\mathcal{S}$ generated by x_μ, $\mu = 1, 2, \ldots, n$. The isomorphism Ω_ϕ depends on the realization ϕ and is given as follows. Let 1 denote the unit in $\mathcal{S}$ ($\mathcal{S}$ is isomorphic to the Fock space built on the vacuum vector $|0\rangle \equiv 1$). Then x_μ and ∂_μ act on $f \in \mathcal{S}$ in a natural way by $x_\mu \cdot f = x_\mu f$ and $\partial_\mu \cdot f = \frac{\partial f}{\partial x_\mu}$. In particular,

$$x_\mu \cdot 1 = x_\mu, \qquad \partial_\mu \cdot 1 = 0. \tag{83}$$

For a monomial $\hat{f}(\hat{x}) \in \mathcal{U}(\mathfrak{g})$ we define

$$\Omega_\phi(\hat{f}(\hat{x})) = \hat{f}(\hat{x}) \cdot 1 \equiv f(x), \tag{84}$$

and extend Ω_ϕ linearly to $\mathcal{U}(\mathfrak{g})$. The map Ω_ϕ is evaluated at $\hat{f}(\hat{x})$ by using the realization (3) and action (83). For example,

$$\Omega_\phi(\hat{x}_\mu) = \sum_\alpha (x_\alpha \phi_{\alpha\mu}(\partial)) \cdot 1 = x_\mu \tag{85}$$

since $\phi_{\alpha\mu}(\partial) = \delta_{\alpha\mu} + o(\partial)$. Similarly, for monomials of order 2 we have

$$\Omega_\phi(\hat{x}_\mu \hat{x}_\nu) = x_\mu x_\nu + \sum_\alpha x_\alpha \frac{\partial \phi_{\alpha\mu}}{\partial \partial_\nu} \cdot 1, \tag{86}$$

where $\frac{\partial \phi_{\alpha\mu}}{\partial \partial_\nu} \cdot 1$ is a first-order coefficient in the Taylor expansion of $\phi_{\alpha\mu}(\partial)$. In general, $\Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_m})$ is a polynomial in the variables $x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu_m}$ whose coefficients are given by the Taylor expansion of $\phi_{\mu\nu}$. The computation of $\Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_m})$ can be done using a recursive formula. Suppose that

$$\Omega_\phi(\hat{x}_{\mu_2} \hat{x}_{\mu_3} \cdots \hat{x}_{\mu_m}) = p(x_{\mu_2}, x_{\mu_3}, \ldots, x_{\mu_m}). \tag{87}$$

Then

$$\begin{aligned} \Omega_\phi(\hat{x}_{\mu_1} \hat{x}_{\mu_2} \cdots \hat{x}_{\mu_m}) &= x_{\mu_1} p(x_{\mu_2}, x_{\mu_3}, \ldots, x_{\mu_m}) \\ &\quad + \sum_\alpha x_\alpha [\phi_{\alpha\mu_1}, p(x_{\mu_2}, x_{\mu_3}, \ldots, x_{\mu_m})] \cdot 1. \end{aligned} \tag{88}$$

The commutator in the above expression is calculated according to

$$[\phi_{\alpha\mu}, x_1 x_2 \cdots x_k] = [\phi_{\alpha\mu}, x_1] x_2 \cdots x_k + x_1 [\phi_{\alpha\mu}, x_2] \cdots x_k + \cdots + x_1 \cdots x_{k-1} [\phi_{\alpha\mu}, x_k]. \tag{89}$$

The inverse map Ω_ϕ^{-1} is defined analogously. Let $\hat{1}$ be the unit in $\mathcal{U}(\mathfrak{g})$. Define the action of $\hat{x}_\mu$ on a monomial $\hat{f}(\hat{x}) \in \mathcal{U}(\mathfrak{g})$ by $\hat{x}_\mu \cdot \hat{f}(\hat{x}) = \hat{x}_\mu \hat{f}(\hat{x})$. The action of ∂_μ on $\hat{f}(\hat{x})$ is defined by $\partial_\mu \cdot \hat{1} = 0$ and $\partial_\mu \cdot \hat{f}(\hat{x}) = (\partial_\mu \hat{f}(\hat{x})) \cdot \hat{1}$ where $\partial_\mu \hat{f}(\hat{x})$ is expressed using the commutation relations $[\partial_\mu, \hat{x}_\nu] = \phi_{\mu\nu}(\partial)$. For the lowest order vector we have

$$\hat{x}_\mu \cdot \hat{1} = \hat{x}_\mu, \qquad \partial_\mu \cdot \hat{1} = 0. \tag{90}$$

Then Ω_ϕ^{-1} is given by

$$\Omega_\phi^{-1}(f(x)) = f(x) \cdot \hat{1} \equiv \hat{f}(\hat{x}) \tag{91}$$

where $f(x)\cdot\hat{1}$ is calculated using the realization (6) and relations (90). For example,

$$\Omega_\phi^{-1}(x_\mu) = \sum_\alpha \hat{x}_\alpha \phi_{\alpha\mu}^{-1}(\partial)\cdot\hat{1} = \hat{x}_\mu \tag{92}$$

since $\phi_{\alpha\mu}^{-1}(\partial) = \delta_{\alpha\mu} + o(\partial)$, and for monomials of order 2 we have

$$\Omega_\phi^{-1}(x_\mu x_\nu) = \hat{x}_\mu\hat{x}_\nu + \sum_\alpha \hat{x}_\alpha \frac{\partial\phi_{\alpha\mu}^{-1}}{\partial\partial_\nu}\cdot\hat{1}. \tag{93}$$

One can show that the right-hand side of equation (93) is invariant under the transposition of indices $\mu \leftrightarrow \nu$, hence $\Omega_\phi^{-1}(x_\mu x_\nu)$ is well defined. Clearly, Ω_ϕ and Ω_ϕ^{-1} can be readily extended to $\overline{\mathcal{U}(\mathfrak{g})}$ and $\overline{\mathcal{S}}$, the formal completions of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{S}$. The star-product of $f, g \in \overline{\mathcal{S}}$ is defined by

$$(f \star_\phi g)(x) = (\hat{f}(\hat{x})\hat{g}(\hat{x}))\cdot 1, \tag{94}$$

where $\hat{f}(\hat{x}) = \Omega_\phi^{-1}(f(x))$ and $\hat{g}(\hat{x}) = \Omega_\phi^{-1}(g(x))$. In the limit as the deformation parameter $a \to 0$ the star-product reduces to ordinary product of functions (cf equation (4)). The star-product on the kappa-deformed space was discussed in [26, 28, 29]; see also [52].

Equation (94) defines the star-product of zero-forms. Following the ideas outlined above we want to extend the star-product to differential forms of arbitrary degree. Our strategy is to associate with ω^k a noncommutative form $\hat{\omega}^k$ such that $\hat{\omega}^k\cdot 1 = \omega^k$ and define the star-product by

$$\omega^k \star_\phi \eta^l = (\hat{\omega}^k\hat{\eta}^l)\cdot 1. \tag{95}$$

It turns out that the star-product (95) is well defined provided the commutator $[\xi_\mu, \hat{x}_\nu]$ is closed in the space of one-forms ξ_μ alone. It depends only on the realizations of the coordinates $\hat{x}_\mu$, hence we also denote it by $\star_\phi$.

First let us consider the star-product of constant forms. Recall that the noncommutative one-form ξ_μ is defined by $\xi_\mu = \sum_\alpha \mathrm{d}x_\alpha h_{\alpha\mu}(\partial)$ where $h_{\alpha\mu}$ satisfies equation (14). The matrix $[h_{\mu\nu}]$ is invertible, hence there is a dual relation $\mathrm{d}x_\mu = \sum_\alpha \xi_\alpha h_{\alpha\mu}^{-1}(\partial)$. Since $h_{\alpha\mu}(\partial)$ is a power series of the type (4), and $\mathrm{d}x_\mu$ and ∂_ν commute, we have

$$\left(\xi_{\mu_1}\xi_{\mu_2}\cdots\xi_{\mu_k}\right)\cdot 1 = \mathrm{d}x_{\mu_1}\,\mathrm{d}x_{\mu_2}\cdots\mathrm{d}x_{\mu_k}. \tag{96}$$

Therefore, to a k-form $\omega^k = \mathrm{d}x_{\mu_1}\,\mathrm{d}x_{\mu_2}\cdots\mathrm{d}x_{\mu_k}$ we associate a unique noncommutative form $\hat{\omega}^k = \xi_{\mu_1}\xi_{\mu_2}\cdots\xi_{\mu_k}$ satisfying $\hat{\omega}^k\cdot 1 = \omega^k$. The star-product of $\omega^k = \mathrm{d}x_{\mu_1}\,\mathrm{d}x_{\mu_2}\cdots\mathrm{d}x_{\mu_k}$ and $\eta^l = \mathrm{d}x_{\nu_1}\,\mathrm{d}x_{\nu_2}\cdots\mathrm{d}x_{\nu_l}$ is trivially given by

$$\omega^k \star_\phi \eta^l = \left(\xi_{\mu_1}\xi_{\mu_2}\cdots\xi_{\mu_k}\xi_{\nu_1}\xi_{\nu_2}\cdots\xi_{\nu_l}\right)\cdot 1. \tag{97}$$

In view of equation (96) the star-product of constant forms is undeformed,

$$\omega^k \star_\phi \eta^l = \omega^k\eta^l, \tag{98}$$

and graded commutative,

$$\omega^k \star_\phi \eta^l = (-1)^{kl}\eta^l \star_\phi \omega^k. \tag{99}$$

Now suppose that ω^k is a general k-form $\omega^k = p(x)\,\mathrm{d}x_{\sigma_1}\,\mathrm{d}x_{\sigma_2}\cdots\mathrm{d}x_{\sigma_k}$ where $p(x)$ is a monomial in x_μ. Then the associated noncommutative form is given by $\hat{\omega}^k = \omega^k\cdot\hat{1}$ where we define $\xi_\mu\cdot\hat{1} = \xi_\mu$. This yields

$$\hat{\omega}^k = \Omega_\phi^{-1}(p(x))\xi_{\sigma_1}\xi_{\sigma_2}\ldots\xi_{\sigma_k}. \tag{100}$$

Indeed, let us denote $\hat{p}(\hat{x}) = \Omega_\phi^{-1}(p(x))$. Using commutativity of $\mathrm{d}x_\mu$ with x_μ and ∂_μ we obtain

$$\hat{\omega}_k = \sum_{\rho_1,\ldots,\rho_k} \mathrm{d}x_{\rho_1}\,\mathrm{d}x_{\rho_2}\cdots\mathrm{d}x_{\rho_k}\,\hat{p}(\hat{x})h_{\rho_1\sigma_1}h_{\rho_2\sigma_2}\cdots h_{\rho_k\sigma_k} \tag{101}$$

$$= \mathrm{d}x_{\sigma_1}\,\mathrm{d}x_{\sigma_2}\cdots\mathrm{d}x_{\sigma_k}\,(\hat{p}(\hat{x}) + o(\partial)). \tag{102}$$

Thus,

$$\hat{\omega}_k \cdot 1 = p(x)\,\mathrm{d}x_{\sigma_1}\,\mathrm{d}x_{\sigma_2}\ldots\mathrm{d}x_{\sigma_k} = \omega^k \tag{103}$$

since $\hat{p}(\hat{x}) \cdot 1 = p(x)$. We note that $\hat{\omega}^k$ given by equation (100) is a unique noncommutative form (up to reordering of $\hat{x}_\mu$ in $\hat{p}(\hat{x})$ using the commutation relations (1)) with the property $\hat{\omega}^k \cdot 1 = \omega^k$ in which the NC coordinates are naturally ordered to the left of ξ_μ. If $\omega^k = p(x)\,\mathrm{d}x_{\mu_1}\,\mathrm{d}x_{\mu_2}\cdots\mathrm{d}x_{\mu_k}$ and $\eta^l = q(x)\,\mathrm{d}x_{\nu_1}\,\mathrm{d}x_{\nu_2}\cdots\mathrm{d}x_{\nu_l}$, then equations (95) and (100) yield

$$\omega^k \star_\phi \eta^l = \big(\hat{p}(\hat{x})\xi_{\mu_1}\cdots\xi_{\mu_k}\hat{q}(\hat{x})\xi_{\nu_1}\cdots\xi_{\nu_l}\big) \cdot 1, \tag{104}$$

where $\hat{p}(\hat{x}) = \Omega_\phi^{-1}(p(x))$ and $\hat{q}(\hat{x}) = \Omega_\phi^{-1}(q(x))$. The star-product (104) is not graded commutative since $\hat{x}_\mu$ and ξ_μ do not commute. The product is well defined provided the commutators $[\xi_\mu, \hat{x}_\nu]$ are closed in the space of one-forms ξ_μ. In this case one can use the commutation relations between ξ_μ and $\hat{x}_\nu$ to write (104) in the natural order with $\hat{x}_\mu$ to the left of ξ_μ and evaluate the star-product using $\big(\hat{p}(\hat{x})\xi_{\mu_1}\cdots\xi_{\mu_k}\big)\cdot 1 = p(x)\,\mathrm{d}x_{\mu_1}\cdots\mathrm{d}x_{\mu_k}$. In view of earlier considerations, the extended star-product can be defined in the covariant left, right and noncovariant realizations discussed in sections 3 and 4. We note that the extended star-product is associative since this property is inherited from associativity of operator multiplication in the superalgebra $\mathcal{A}$.

Finally, let us consider the exterior derivative acting on the star-product of forms. In the realization of type I the exterior derivative is undeformed, $\hat{d} = d \equiv \sum_\alpha \mathrm{d}x_\alpha \partial_\alpha$. Then one can show that

$$\mathrm{d}\omega = (\hat{d}\hat{\omega}) \cdot 1, \tag{105}$$

where $\hat{\omega} \cdot 1 = \omega$. Using the star-product (95) and Leibniz rule (17) one finds

$$\mathrm{d}(\omega \star_\phi \eta) = \mathrm{d}\omega \star_\phi \eta + (-1)^{|\omega|}\omega \star_\phi \mathrm{d}\eta. \tag{106}$$

Hence, in type I realization the Leibniz rule for the extended star-product is undeformed. It would be interesting to investigate the action of the induced exterior derivative on the star-product of forms in other realizations when $\hat{d}$ is given by a general expression (11).

6. Concluding remarks

In this paper, we have investigated the differential algebra of forms on the kappa-deformed space. Our construction of the exterior derivative $\hat{d}$ and one-forms ξ_μ is based on the realizations of NC coordinates $\hat{x}_\mu$ in terms of formal power series in the Weyl algebra. We have shown that for each realization of $\hat{x}_\mu$ there is an infinite family of the exterior derivatives $\hat{d}$ which uniquely determine the one-forms ξ_μ. The exterior derivative is a nilpotent operator and it satisfies the undeformed Leibniz rule. The NC coordinates $\hat{x}_\mu$, derivatives ∂_μ and one-forms ξ_μ generate a $\mathbb{Z}_2$-graded algebra. The subalgebra generated by $\hat{x}_\mu$ and ∂_μ is a deformed Heisenberg algebra. The algebra generated by $\hat{x}_\mu$ and ξ_μ is generally not closed under the commutator bracket since $[\xi_\mu, \hat{x}_\nu]$ may involve an infinite series in ∂_μ. Only in special cases of the covariant left, right and noncovariant realizations the algebra is closed under the commutator bracket. Furthermore, the commutator $[\xi_\mu, \hat{x}_\nu]$ is nonzero in all realizations. For higher order forms we have shown that the exterior derivative satisfies the graded Leibniz rule, and the graded Jacobi identity also holds. However, the graded commutativity law holds only for $\hat{x}_\mu$-independent forms. In the limit when the deformation parameter $a \to 0$ our theory reduces to classical results.

The exterior derivative and one-forms have been analyzed in both covariant and noncovariant realizations. In the covariant case we have found explicit representations of $\hat{d}$ and ξ_μ in the left, right and natural realizations. We have also found a closed form expression for the commutator $[\xi_\mu, \hat{x}_\nu]$ in these realizations, and derived an approximation to first order in a in the natural realization. In the noncovariant case we have constructed a one-parameter family of realizations of $\hat{d}$ and ξ_μ. For this family of realizations the commutator $[\xi_\mu, \hat{x}_\nu]$ is always closed in the space of one-forms ξ_μ.

We have also extended the star-product from zero-forms to differential forms of arbitrary degree. The star-product can be defined for realizations in which $[\xi_\mu, \hat{x}_\nu]$ is closed in the space of one-forms ξ_μ. It depends only on the realizations of both the NC coordinates $\hat{x}_\mu$. For differential forms with constant coefficients the star-product is undeformed and graded commutative, but for arbitrary forms this is no longer true. It was shown that the exterior derivative acting on the extended star-product satisfies the undeformed Leibniz rule in type I realization. It would be interesting to investigate possible relations between our approach to the star-product of differential forms and the recent work presented in [51].

Finally, the notion of the twist operator is very important in the construction of the star-product from both the mathematical [53, 54] and physical [55–59] points of view. The twist operator for zero-forms on the kappa-deformed space was constructed in [30] and [59], and was also considered in [27]. However, it remains an open problem to see if there exists a twist operator that leads to the star-product of differential forms defined in this work.

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Appendix A

In this appendix, we find the solution of the system of equations (47) and (48). Let us write equation (47) in equivalent forms as

$$(AZ^{-1} - B)\frac{\partial H_1}{\partial A} + 2B\sqrt{1-B}\frac{\partial H_1}{\partial B} + (Z^{-1} - 1)H_1 = 0. \tag{A.1}$$

We assume that H_1 can be factored as $H_1(A, B) = ZF_1(B)$ which leads to the following differential equation for F_1,

$$2B\sqrt{1-B}F_1'(B) + \left(\sqrt{1-B} - 1\right)F_1(B) = 0. \tag{A.2}$$

The boundary condition for H_1 implies that $\lim_{a\to 0} F_1(B) = 1$. Now the solution to equation (A.2) is readily found to be

$$F_1(B) = \frac{2(1-\sqrt{1-B})}{B}, \tag{A.3}$$

hence

$$H_1(A, B) = \frac{2(1-\sqrt{1-B})}{B(-A+\sqrt{1-B})}. \tag{A.4}$$

Next, let us consider equation (48) which we write equivalently as

$$(AZ^{-1}-B)\frac{\partial H_2}{\partial A}+2B\sqrt{1-B}\frac{\partial H_2}{\partial B}+(2\sqrt{1-B}-1)H_2=-H_1. \tag{A.5}$$

We apply a similar method of 'separation of variables' assuming that $H_2(A,B)=ZF_2(B)+F_3(B)$. Inserting the ansatz for H_1 and H_2 into equation (A.5), and grouping the terms depending only on B on the right-hand side, we obtain

$$\begin{aligned}&AF_2(B)+Z^{-1}(2B\sqrt{1-B}F_3'(B)+(2\sqrt{1-B}-1)F_3(B))\\&\qquad=-2B\sqrt{1-B}F_2'(B)-(2\sqrt{1-B}-1)F_2(B)-F_1(B).\end{aligned} \tag{A.6}$$

Let us define the function

$$G(B)=2B\sqrt{1-B}F_3'(B)+\big(2\sqrt{1-B}-1\big)F_3(B). \tag{A.7}$$

Then the variables in equation (A.6) can be separated as

$$A(F_2(B)-G(B))=-2B\sqrt{1-B}F_2'(B)-(2\sqrt{1-B}-1)F_2(B)-F_1(B)-\sqrt{1-B}G(B). \tag{A.8}$$

We conclude that both sides of the equation must be zero which implies that F_2 and F_3 satisfy the following system of differential equations:

$$2B\sqrt{1-B}F_2'(B)+(3\sqrt{1-B}-1)F_2(B)=-F_1(B), \tag{A.9}$$

$$2B\sqrt{1-B}F_3'(B)+(2\sqrt{1-B}-1)F_3(B)=F_2(B). \tag{A.10}$$

Using the boundary condition for H_2 we find that in the limit $a\to 0$ both $F_2(B)$ and $F_3(B)$ must be finite. Taking this into account, integration of the system (A.9) and (A.10) yields

$$F_2(B)=F_3(B)=-2\left(\frac{1-\sqrt{1-B}}{B}\right)^2. \tag{A.11}$$

Therefore,

$$H_2(A,B)=-2(1-A+\sqrt{1-B})\left(\frac{1-\sqrt{1-B}}{B}\right)^2. \tag{A.12}$$

Appendix B

In this appendix we give a brief derivation of the result (65). We shall do this in two steps. First we calculate the commutator $[\xi_\mu,\hat{x}_\nu]$ where $\xi_\mu=\sum_\alpha \mathrm{d}x_\alpha h_{\alpha\mu}(\partial)$ and $\hat{x}_\nu$ is given in the natural realization (30). We have

$$[\xi_\mu,\hat{x}_\nu]=Z^{-1}\sum_\alpha[h_{\alpha\mu},x_\nu]\mathrm{d}x_\alpha+\partial_\nu\sum_\alpha[h_{\alpha\mu},\mathrm{i}ax]\,\mathrm{d}x_\alpha. \tag{B.1}$$

Expressing $h_{\alpha\mu}$ by equation (56) and making use of

$$\frac{\partial f(A,B)}{\partial\partial_\mu}=\mathrm{i}\frac{\partial f}{\partial A}a_\mu+2a^2\frac{\partial f}{\partial B}\partial_\mu, \tag{B.2}$$

after some manipulation we find

$$\begin{aligned}\sum_\alpha[h_{\alpha\mu},x_\nu]\mathrm{d}x_\alpha&=\left(\mathrm{i}\frac{\partial L_1}{\partial A}a_\nu+2a^2\frac{\partial L_1}{\partial B}\partial_\nu\right)\mathrm{d}x_\mu\\&\quad+(\mathrm{i}L_3a_\mu+a^2L_4\partial_\mu)\,\mathrm{d}x_\nu+\mathrm{i}S_{\mu\nu}(a\,\mathrm{d}x)+T_{\mu\nu}(\partial\,\mathrm{d}x),\end{aligned} \tag{B.3}$$

where we have defined

$$S_{\mu\nu} = L_2\delta_{\mu\nu} + 2\left(B\frac{\partial L_5}{\partial B} + L_5\right)\,\mathrm{i}a_\mu\partial_\nu + \frac{\partial L_2}{\partial A}\,\mathrm{i}a_\nu\partial_\mu + 2a^2\frac{\partial L_2}{\partial B}\partial_\mu\partial_\nu - \partial^2\frac{\partial L_5}{\partial A}a_\mu a_\nu, \tag{B.4}$$

$$T_{\mu\nu} = a^2L_4\delta_{\mu\nu} + 2a^2\frac{\partial L_3}{\partial B}\,\mathrm{i}a_\mu\partial_\nu + a^2\frac{\partial L_4}{\partial A}\,\mathrm{i}a_\nu\partial_\mu + 2a^4\frac{\partial L_4}{\partial B}\partial_\mu\partial_\nu - \frac{\partial L_3}{\partial A}a_\mu a_\nu. \tag{B.5}$$

A similar computation yields

$$\sum_\alpha [h_{\alpha\mu}, \mathrm{i}ax]\,\mathrm{d}x_\alpha = a^2E_1\,\mathrm{d}x_\mu + (\mathrm{i}E_2a_\mu + a^2E_3\partial_\mu)(\mathrm{i}a\,\mathrm{d}x) + a^2(\mathrm{i}E_4a_\mu + a^2E_5\partial_\mu)(\partial\,\mathrm{d}x) \tag{B.6}$$

where the functions E_i are defined by

$$E_1 = 2A\frac{\partial L_1}{\partial B} - \frac{\partial L_1}{\partial A}, \tag{B.7}$$

$$E_2 = L_2 + L_3 + 2AL_5 + 2AB\frac{\partial L_5}{\partial B} - B\frac{\partial L_5}{\partial A}, \tag{B.8}$$

$$E_3 = L_4 + 2A\frac{\partial L_2}{\partial B} - \frac{\partial L_2}{\partial A}, \tag{B.9}$$

$$E_4 = L_4 + 2A\frac{\partial L_3}{\partial B} - \frac{\partial L_3}{\partial A}, \tag{B.10}$$

$$E_5 = 2A\frac{\partial L_4}{\partial B} - \frac{\partial L_4}{\partial A}. \tag{B.11}$$

Combining equations (B.3) and (B.6) we obtain

$$[\xi_\mu, \hat{x}_\nu] = \mathrm{d}x_\mu P^{(1)}_\nu + \mathrm{d}x_\nu P^{(2)}_\mu(\partial) + (\mathrm{i}a\,\mathrm{d}x)Q^{(1)}_{\mu\nu} + (\partial\mathrm{d}x)Q^{(2)}_{\mu\nu}. \tag{B.12}$$

where the functions $P^{(i)}_\mu$ and $Q^{(i)}_{\mu\nu}$ are given by

$$P^{(1)}_\nu = Z^{-1}\frac{\partial L_1}{\partial A}\,\mathrm{i}a_\nu + a^2\left(2Z^{-1}\frac{\partial L_1}{\partial B} + E_1\right)\partial_\nu, \tag{B.13}$$

$$P^{(2)}_\mu = Z^{-1}L_3\,\mathrm{i}a_\mu + a^2Z^{-1}L_4\partial_\mu, \tag{B.14}$$

$$Q^{(1)}_{\mu\nu} = Z^{-1}S_{\mu\nu} + E_2\,\mathrm{i}a_\mu\partial_\nu + a^2E_3\partial_\mu\partial_\nu, \tag{B.15}$$

$$Q^{(2)}_{\mu\nu} = Z^{-1}T_{\mu\nu} + a^2E_4\,\mathrm{i}a_\mu\partial_\nu + a^4E_5\partial_\mu\partial_\nu. \tag{B.16}$$

In the second step, we wish to express the commutator (B.12) in terms of the one-forms ξ_μ and derivatives ∂_μ. In order to replace $\mathrm{d}x_\mu$ by ξ_μ we write $\mathrm{d}x_\mu = \sum_\alpha h^{-1}_{\alpha\mu}(\partial)\xi_\mu$ where $h^{-1}_{\alpha\mu}$ is the inverse of the matrix $h_{\alpha\mu}$. The inverse matrix should have the same index structure as $h_{\alpha\mu}$, hence we look for $h^{-1}_{\alpha\mu}$ in the form

$$h^{-1}_{\alpha\mu}(\partial) = G_1\delta_{\alpha\mu} + \mathrm{i}G_2a_\alpha\partial_\mu + \mathrm{i}G_3a_\mu\partial_\alpha + a^2G_4\partial_\alpha\partial_\mu - \partial^2G_5a_\alpha a_\mu. \tag{B.17}$$

The condition $\sum_\alpha h_{\alpha\beta}h^{-1}_{\beta\mu} = \delta_{\alpha\mu}$ implies that the functions G_k satisfy the following system of equations:

$$G_1 = L_1^{-1}, \tag{B.18}$$

$$-(L_1 + AL_2 - BL_5)G_2 - B(L_2 + AL_5)G_4 = L_2L_1^{-1}, \tag{B.19}$$

$$(L_3 - AL_4)G_2 - (L_1 + AL_3 + BL_4)G_4 = L_4L_1^{-1}, \tag{B.20}$$

$$-(L_1 + AL_3 + BL_4)G_3 + B(L_3 - AL_4)G_5 = L_3L_1^{-1}, \tag{B.21}$$

$$-(L_2 + AL_5)G_3 - (L_1 + AL_2 - BL_5)G_5 = L_5L_1^{-1}. \tag{B.22}$$

The solution of the system is given by

$$G_2 = \frac{1}{M}[-(L_1 + AL_3 + BL_4)L_2 + B(L_2 + AL_5)L_4], \tag{B.23}$$

$$G_3 = -\frac{1}{M}[(L_1 + AL_2 - BL_5)L_3 + B(L_3 - AL_4)L_5], \tag{B.24}$$

$$G_4 = -\frac{1}{M}[(L_3 - AL_4)L_2 + (L_1 + AL_2 - BL_5)L_4], \tag{B.25}$$

$$G_5 = \frac{1}{M}[(L_2 + AL_5)L_3 - (L_1 + AL_3 + BL_4)L_5], \tag{B.26}$$

where

$$M = L_1[(L_1 + AL_2 - BL_5)(L_1 + AL_3 + BL_4) + B(L_2 + AL_5)(L_3 - AL_4)]. \tag{B.27}$$

Now, with the functions G_k defined as above, we have

$$\mathrm{d}x_\mu = \sum_\alpha h^{-1}_{\alpha\mu}(\partial)\xi_\alpha = G_1\xi_\mu + (\partial^2 G_5\, \mathrm{i}a_\mu + G_2\partial_\mu)(\mathrm{i}a\xi) + (G_3\mathrm{i}a_\mu + a^2G_4\partial_\mu)(\partial\xi). \tag{B.28}$$

Using equation (B.28) to eliminate $\mathrm{d}x_\mu$ from the commutator (B.12) we obtain

$$[\xi_\mu, \hat{x}_\nu] = \xi_\mu\frac{P^{(2)}_\nu}{L_1} + \xi_\nu\frac{P^{(2)}_\mu}{L_1} + (\mathrm{i}a\xi)R^{(1)}_{\mu\nu} + (\partial\xi)R^{(2)}_{\mu\nu}, \tag{B.29}$$

where $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are defined by

$$\begin{aligned} R^{(1)}_{\mu\nu} = {} & \partial^2G_5\big(P^{(1)}_\nu\, \mathrm{i}a_\mu + P^{(2)}_\mu\, \mathrm{i}a_\nu\big) + G_2\big(P^{(1)}_\nu\partial_\mu + P^{(2)}_\mu\partial_\nu\big) \\ & + (G_1 + AG_2 - BG_5)Q^{(1)}_{\mu\nu} + \partial^2(G_2 + AG_5)Q^{(2)}_{\mu\nu}, \end{aligned} \tag{B.30}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = {} & G_3\big(P^{(1)}_\nu\mathrm{i}a_\mu + P^{(2)}_\mu\mathrm{i}a_\nu\big) + a^2G_4\big(P^{(1)}_\nu\partial_\mu + P^{(2)}_\mu\partial_\nu\big) \\ & + a^2(AG_4 - G_3)Q^{(1)}_{\mu\nu} + (G_1 + AG_3 + BG_4)Q^{(2)}_{\mu\nu}. \end{aligned} \tag{B.31}$$

Tracing back the computations we can express the commutator (B.29) explicitly in terms of $L_1, \ldots, L_5$ and their partial derivatives, but the expressions are cumbersome and not useful for practical calculations.

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